

# **Heat and Mass transfer**

## **Heat Transfer from Extended Surfaces**

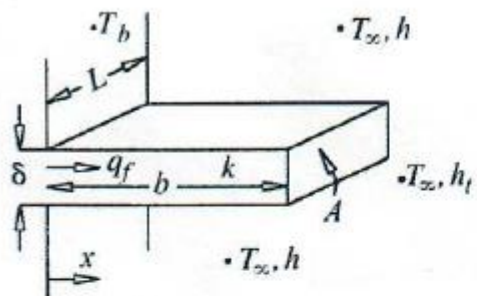
## **Heat Transfer from Extended Surface**

The heat transfer rate from a surface by convection process is proportional to the surface area, the heat transfer coefficient and the temperature difference between the surface and the fluid

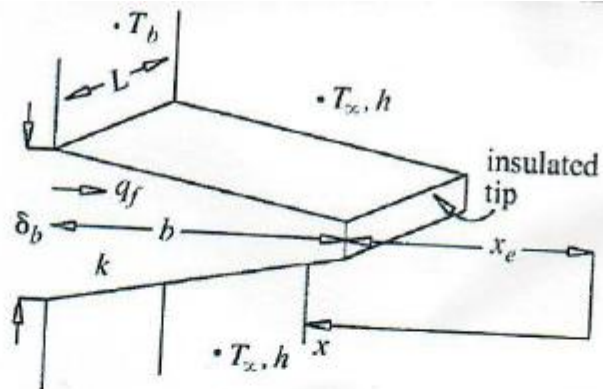
- Temperature difference and the heat transfer coefficient are fixed
- Surface area is to be increased artificially so as to increase the heat transfer.

These extended surfaces are called fins or spines and their purpose is to increase the heat removal between the surface and the surrounding ambient fluid (flowing or static).

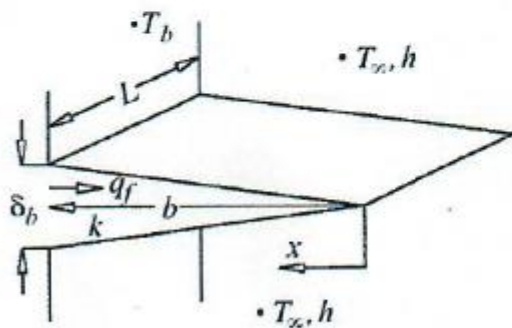
Some of these fins can be seen on air cooled engines, attached to car radiator, condenser of refrigerators, boiler tubes and many other heat exchanger devices.



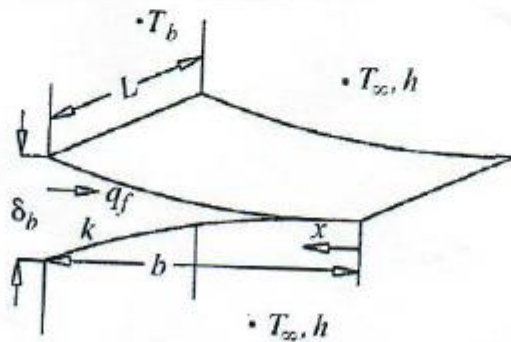
Rectangular



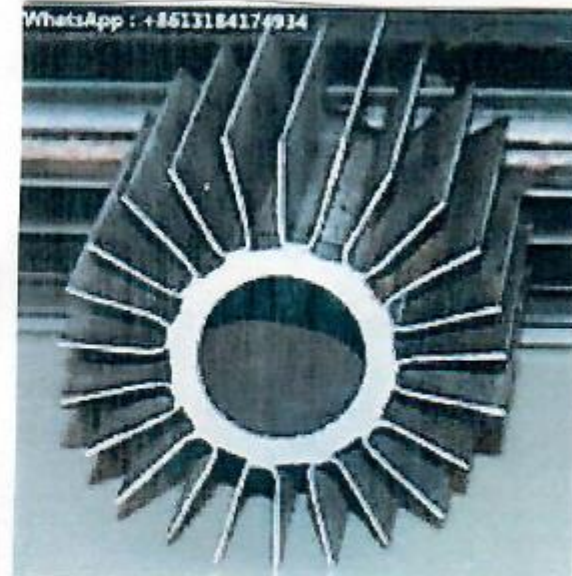
Trapezoidal

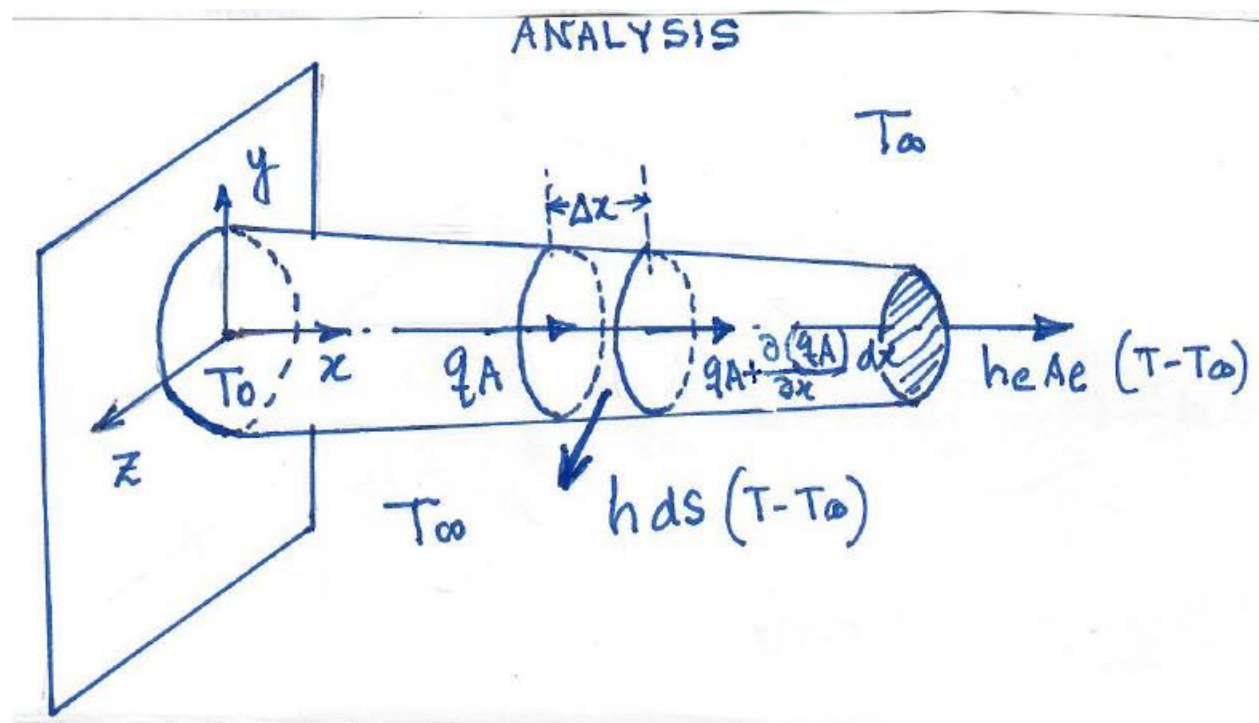


Triangular



Concave





Following are the considerations:

1. Steady state heat transfer 2. Conductivity,  $k$  of the fin is constant. 3. Heat transfer within the solid is in the  $x$  direction only (This is possible only if the area of cross section is small compared to the fin length and the boundary conditions are constant with respect to  $z$  and  $y$ ). Another implication of the third assumption is that the temperature will be uniform across a cross section of the fin.

$A(x)$ : Area of cross section;  $S(x)$  Surface area of the fin from  
 $x = 0$ , to  $x$

$k$ : conductivity of the material,  $h$  = heat transfer coefficient of  
the fin surface

Heat transfer by conduction to the left face of the element =  $qA$  heat transfer by conduction out of the right face =  $qA + \frac{\partial}{\partial x}(qA)\Delta x$

Heat transfer by convection from the surface  $\delta S$  of the element exposed to the fluid at a temperature  $T_\infty = h \delta S (T_{avg} - T_\infty)$

Where  $T_{avg}$  is the temperature of the element (average):

1<sup>st</sup> law of thermodynamic yields

$$qA = qA + \frac{\partial}{\partial x}(qA)\Delta x + h\delta S(T_{avg} - T_\infty)$$

$$\text{or, } \frac{\partial}{\partial x}(qA) + h \frac{\delta S}{\Delta x}(T_{avg} - T_\infty) = 0$$

$$\text{In the limit } \Delta x \rightarrow 0 \quad : \quad \frac{\delta S}{\Delta x} = \frac{dS}{dx} \quad \text{and} \quad T_{avg} \rightarrow T(x)$$

$$\text{or, } A \frac{dq}{dx} + q \frac{dA}{dx} + h \frac{dS}{dx}(T - T_\infty) = 0$$

Substituting  $q = -k \frac{dT}{dx}$

$$\frac{d^2T}{dx^2} + \frac{dA/dx}{A} \frac{dT}{dx} - \frac{h}{kA} \frac{dS}{dx} (T - T_\infty) = 0$$

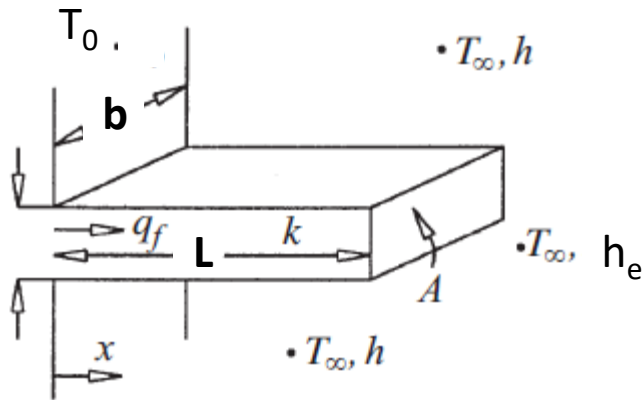
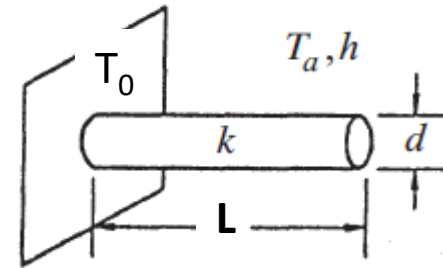
To find a unique solution of this equation two boundary condition are required. At  $x=0$  the fin will have the temperature of the base metal unless there is some contact resistance at this point.

$$at\ x = 0 : T = T_0 \quad : \quad at\ x = L \quad : \quad h_e A_e (T - T_\infty) = -k A_e \frac{dT}{dx}$$

## Fins having same cross section all along its length

This types of fins amenable to simple mathematical analysis.

Rectangular, as well as circular or other constant cross section fins can be analysed by this procedure



L= Length of the fin

$b$  = width,  $t$  = thickness,  $A = bt$

$$P = 2(b + t), \quad S = Px$$

$$\frac{dA}{dx} = 0, \quad \text{and} \quad \frac{dS}{dx} = P$$

$r$  = radius,  $A = \pi r^2$

$$P = 2\pi r, \quad S = Px$$

$$\frac{dA}{dx} = 0, \quad \text{and} \quad \frac{dS}{dx} = P$$

We can write

$$\frac{d^2 T}{dx^2} - \frac{hP}{kA} (T - T_{\infty}) = 0$$

Substituting  $\theta = (T - T_{\infty})$  and  $m^2 = \frac{hP}{kA}$  we get

$$\frac{d^2 \theta}{dx^2} - m^2 \theta = 0$$

The solution of the equation:  $\theta = C_1 e^{-mx} + D_1 e^{mx}$

$$\text{or, } \theta = C_2 \cosh mx + D_2 \sinh mx$$

$$\text{or, } \theta = C_3 \cosh m(L - x) + D_3 \sinh m(L - x)$$

$$\left. \begin{array}{l} \text{at } x = 0, T = T_0 \text{ ; i.e., } \theta = \theta_0 = T_0 - T_\infty \\ \text{at } x = L, -k \frac{dT}{dx} = h_e (T - T_\infty) \text{ or, } -k \frac{d\theta}{dx} = h_e \theta \end{array} \right\} \text{ case 1}$$

Substituting these boundary conditions in

$$\theta = C_3 \cosh m(L - x) + D_3 \sinh m(L - x) \quad \text{we get} \quad \left| \begin{array}{l} \theta_0 = C_3 \cosh mL + D_3 \sinh mL \end{array} \right. \quad \text{For } \text{at } x = 0, T = T_0$$

$$-k \left[ -m C_3 \sinh m(L - x) - m D_3 \cosh m(L - x) \right] = h_e \left[ C_3 \cosh m(L - x) + D_3 \sinh m(L - x) \right]$$

$$\text{at } x = L, k D_3 m = h_e C_3 \quad \text{i.e., } D_3 = C_3 \frac{h_e}{k m} = C_3 H$$

$$C_3 \left[ \cosh mL + H \sinh mL \right] = \theta_0$$

$$C_3 = \frac{\theta_0}{\cosh mL + H \sinh mL} \quad ; \quad D_3 = \frac{\theta_0 H}{\cosh mL + H \sinh mL}$$

$$\theta = \frac{\theta_0}{\cosh mL + H \sinh mL} \cosh m(L - x) + \frac{\theta_0 H}{\cosh mL + H \sinh mL} \sinh m(L - x)$$

We can write

$$\frac{\theta}{\theta_0} = \frac{\cosh m(L-x) + H \sinh m(L-x)}{\cosh mL + H \sinh mL} \text{ where } H = \frac{h_e}{km}$$

$$\text{Now, } q_{fin} = q_{base} = -kA \left. \frac{dT}{dx} \right|_{x=0} = -kA \left. \frac{d\theta}{dx} \right|_{x=0}$$

$$q_{fin} = \frac{-kA \left[ -m \{ \sinh m(L-x) + H \cosh m(L-x) \} \right]_{x=0}}{\cosh mL + H \sinh mL} \theta_0 \left. \right\}_{at \ x=0}$$

$$q_{fin} = +kA \sqrt{\frac{hP}{kA}} \cdot \theta_0 [(\sinh mL + H \cosh mL) / (\cosh mL + H \sinh mL)]$$

$$q_{fin} = \sqrt{kAhP} \theta_0 [(\sinh mL + H \cosh mL) / (\cosh mL + H \sinh mL)]$$

Case-1 Convective heat transfer

$$\text{at } x=L: -k \left. \frac{d\theta}{dx} \right|_{x=L} h_e \theta$$

Case-2

Infinite fin at  $x = \infty$ ,  $\theta = 0$

Case-3 Insulated tip fin

$$\text{at } x=L, k \frac{d\theta}{dx} = 0$$

For fins with insulated tips (case 3)

In the earlier solution, let us substitute  $H=0$

$$D_3 = 0; \quad C_3 \frac{\theta_0}{\cosh mL}; \quad \theta = \theta_0 \frac{\cosh m(L-x)}{\cosh mL}$$

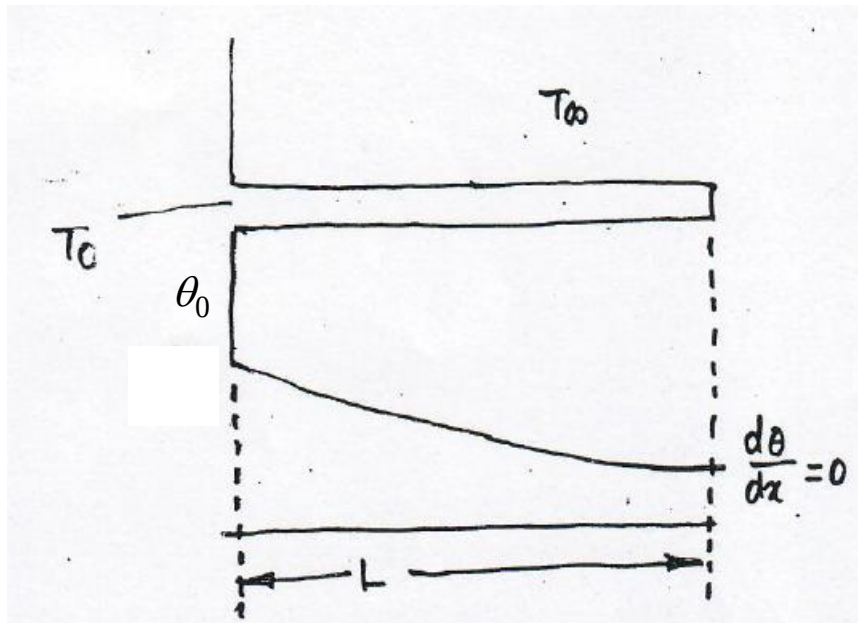
$$\theta = \theta_0 \left[ \frac{e^{m(L-x)} + e^{-m(L-x)}}{2 \cosh mL} \right]$$

$$\text{or, } \frac{\theta}{\theta_0} = \frac{2 \cosh m(L-x)}{2 \cosh mL}$$

$$q_{fin} = q_{base} = -kA \frac{dT}{dx} \Big|_{x=0} = -kA \frac{d\theta}{dx} \Big|_{x=0}$$

$$= -kA \left[ \frac{-m \{ 2 \sinh m(L-x) \}}{2 \cosh mL} \right] \theta_0$$

$$= +kAm \theta_0 \left[ \tanh m(L-x) \right] = \sqrt{kAhP} \theta_0 \tanh mL$$



| Case | Tip Condition                                                                             | Temperature Distribution                                                                               | Fin Heat Transfer rate $q_f$                                                              |
|------|-------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------|
| A    | Convective heat transfer<br>at $x = L$ : $-k \frac{d\theta}{dx} \Big _{x=L} = h_e \theta$ | $\frac{\theta}{\theta_0} = \left[ \frac{\cosh m(L-x) + H \sinh m(L-x)}{\cosh mL + H \sinh mL} \right]$ | $\sqrt{hPkA} \theta_0 \left[ \frac{\sinh mL + H \cosh mL}{\cosh mL + H \sinh mL} \right]$ |
| B    | Infinite fin at $x = \infty$ , $\theta = 0$                                               | $\theta = \theta_0 e^{-mx}$                                                                            | $\sqrt{hPkA} \theta_0$                                                                    |
| C    | Insulated tip fin<br>at $x = L$ , $k \frac{d\theta}{dx} = 0$                              | $\frac{\theta}{\theta_0} = \left[ \frac{e^{m(L-x)} + e^{-m(L-x)}}{2 \cosh mL} \right]$                 | $\sqrt{hPkA} \theta_0 \tanh mL$                                                           |

## Efficiency of a Fin

There is one criterion which is called effectiveness of a fin

$$\varepsilon = \frac{-k \left. \frac{dT}{dx} \right|_{base}}{h(T_0 - T_\infty)} = \frac{\text{heat transfer rate from the fin}}{\text{heat transfer rate from the base in the absence of fin}}$$

This ratio is greater than unity

The second criterion which is used as an index of performance of the fin is called the fin efficiency

$$\eta_f = \frac{\text{heat transfer rate from the fin surface}}{\text{heat transfer rate which would occur if the entire fin surface was at the base temperature } T_0}$$

$$= \frac{Q}{Q_0} = \frac{\int_0^s h(T - T_\infty) ds}{h(T_0 - T_\infty)S} = \frac{\int_0^s (T - T_\infty) ds}{(T_0 - T_\infty)S}$$

Efficiency of a straight fin (with insulated tip boundary condition)

$$Q_0 = h \{ (2bL) + (2Lt) + bt \} \theta_0 = 2bLh \theta_0 \left( 1 + \frac{t}{b} + \frac{t}{2L} \right)$$

$$Q = \sqrt{kAhP} \theta_0 \tanh[mL] = \sqrt{kh \, 2(b+t)bt} \theta_0 \tanh mL$$

$$\eta_f = \frac{\sqrt{2kh \, bt(b+t)} \theta_0 \tanh mL}{2bLh \theta_0 (1 + t/b + t/2L)} = \frac{\cancel{b}\sqrt{2kht}}{2\cancel{b}Lh} \tanh mL \frac{\left(1 + \cancel{t}/\cancel{b}\right)^{1/2}}{(1 + t/b + t/2L)}$$

$$\eta_f = \frac{1}{L} \sqrt{\frac{kt}{2h}} \tanh mL \frac{\left(1 + \cancel{t}/\cancel{b}\right)^{1/2}}{\left(1 + \cancel{t}/\cancel{b} + \cancel{t}/2L\right)} \quad \left| \quad \begin{aligned} mL &= \sqrt{\frac{hP}{kA}} L = \sqrt{\frac{h \, 2(b+t)}{k(bt)}} L \\ &= L \sqrt{\frac{2h}{kt}} \left(1 + \cancel{t}/\cancel{b}\right)^{1/2} \end{aligned} \right.$$

$$\eta_f = \frac{\tanh mL}{mL} \cdot \frac{(1+t/b)}{(1+t/b+t/2L)}$$

$$\left(1 + \frac{t}{b} + \frac{t}{2L}\right) = \left(1 + \frac{t}{b}\right) \left\{1 + \frac{t}{2L} \cdot \frac{b}{(b+t)}\right\} \approx (1+t/b)$$

$$\text{so, } \eta_f = \frac{\tanh mL}{mL}$$

### Hyperbolic functions:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\frac{d}{dx}(\sinh x) = \frac{1}{2} \left[ e^x - (-e^{-x}) \right] = \cosh x$$

$$\frac{d}{dx}(\cosh x) = \frac{1}{2} \left[ e^x + (-e^{-x}) \right] = \sinh x$$

*Thank you*